CASIMIR ELEMENTS AND KERNEL OF WEITZENBÖK DERIVATION.

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ABSTRACT. Let $k[X] := k[x_0, x_1, \ldots, x_n]$ be a polynomial algebra over a field k of characteristic zero. We offer an algorithm for calculation of kernel of Weitzenbök derivation $d(x_i) = x_{i-1}, \ldots,$ $d(x_0) = 0$, $i = 1 \ldots n$ that is based on an analogue of the well known Casimir elements of finite dimensional Lie algebras. By using this algorithm, the kernel is calculated in the case $n < 7$.

1. INTRODUCTION

Let $k[X] := k[x_0, x_1, \ldots, x_n]$ be a polynomial algebra over a field k of characteristic zero. For arbitrary derivation D of $k[X]$ denote by $k[X]^D$ a kernel of D, that is,

$$k[X]^D := \mathrm{Ker} D := \{f \in k[X]; D(f) = 0\}.$$

Let d be Weitzenbök derivation defined by rule $d(x_i) = x_{i-1}, \ldots, d(x_0) = 0$, $i = 1 \ldots n$. The kernel of the derivation d was actively studied by various authors. The well-known Weitzenbök's theorem implies a finite generation of the derivation d. A minimal generating set of algebra $k[X]^d$ for $n \le 4$ presented in [1] and in [5] for $n \le 5$. The aim of this paper is to calculate a generating set of the kernel of Weitzenbök derivation in case $n \le 6$. We offer the general description of a kernel of arbitrary polynomialy derivation D by using a construction which is a commutative analogue to a construction of Casimir elements of finite dimension Lie algebras. Let us recal that a (generalised) Casimir element of a finite dimensional Lie algebra L is called an cental element of the universal eneveloping algebra $U(L)$ of the follow form

$$\sum_i u_i u_i^*,$$

where $\{u_i\}$, $\{u_i^*\}$ are a dual bases of a contragradient $L-$ modules in $U(L)$ with respect to the adjoint action of the Lie algebra on the algebra $U(L)$. In the case char(k)=0 it is well known that every element of the center of an universal eneveloping algebra $U(L)$ is a Casimir element. The case char(k)=p> 0 was studied by present autor in [4].

This paper is organized as follows. In section 2 a conceptions of $D-$ module, dual $D-$modules and Casimir elements are introduced. We prove that arbitrary Casimir elements of derivation D belongs to the kernel $k[X]^D$. For any linear derivation D is showed that a theorem inverse to above theorem is true, i.e., any element of kernel $k[X]^D$ is a Casimir element.

In section 3 we study Casimir elements(polynomial) of the Weitzenbök derivation d. Since d is linear derivation we see that the problem of finding of the kernel $k[X]^d$ is equivalent to the problem of finding a realisation of a dual $d-$modules in $k[X]$. We get such realisations from any element of the kernel $k[X]^d$ by using two new derivations e and $\hat{d}$ which arising with the natural embedding of $d-$modules into $\mathfrak{s}l_2-$modules.

In section 4 we introduce a maps $\tau_i : k[X]^d \longrightarrow k[X]^d$. To any element z of kernel derivation d we assign certain family $\tau_i(z)$ elements of the kernel and studying properties of this correspondence.

In section 5 we present a criterion to verify if a subalgebra of $k[X]^d$ coincides with the whole algebra $k[X]^d$. By using this criterion we offer an algorithm for computing the kernel $k[X]^d$.

By using the algoritm in section 6 we compute and present a list of generating elements of the kernel of of Weitzenbök derivation in the case $n < 7$. The result for $n = 6$ is new.

All objects which we use in this paper: rings, algebras, vector spases, isomorphisms, polynomials are considered over field k of a characteristic zero.

2. CASIMIR'S ELEMENTS OF A DERIVATION.

The aim of this section is to offer a method of constructing elements of $k[X]^D$ where D is an arbitrary derivation of $k[X]$.

Definition 2.1. *A vector space V is called a $D-$ module if $D(V) \subseteq V$.*

The derivation d defined by $d(x_n) = x_{n-1}$, $d(x_{n-1}) = x_{n-2},\dots,$ $d(x_0) = 0$ is called the Weitzenbök derivation. Let X_m be a vector space spanned by the elements $x_0, x_1, \dots, x_m$. Then, obviously, X_m is $d-$ module. We denote by D_V a matrix of derivation D in some ordered fixed basis of the space V. For example the matrix of derivation d in X_m is Jordan cell $J_{m+1}(0)$.

Definition 2.2. *$D-$ module V is called dual to $D-$ module V^* if there are dual bases $\{v_i\}$, $\{v_i^*\}$ of V and V^* such that*

$$D_{V^*} = (-D_V)^T.$$

The bases $\{v_i\}$, $\{v_i^*\}$ are also called dual bases. From definition 2.2 it follows that the derivation d acts on X_m^* such that $d(x_i^*) = -x_{i+1}, d(x_m) = 0$.
Let $D, \hat{D}$ be a derivations of $k[X]$.

Definition 2.3. *$D-$ module V and $\hat{D}-$ module W are called isomorphic if there is a linear space isomorphism $\phi : V \longrightarrow W$ such that $D\phi = \phi\hat{D}$. In this case we write $V \cong W$.*

Theorem 2.1. *$X_m^* \cong X_m$ as $d-$ modules.*

Proof. Let $\varphi : X_m^* \longrightarrow X_m$ be the isomorphism defined by rule $\varphi(x_i^*) = (-1)^{i+1}x_{m-i+1}$. Then

$$d(\varphi(x_i^*)) = d((-1)^{i+1}x_{m-i+1}) = (-1)^{i+1}x_{m-i},$$

and

$$\varphi(d(x_i^*)) = \varphi(-x_{i+1}^*) = -(-1)^{i+2}x_{m-i} = (-1)^{i+1}x_{m-i} = d(\varphi(x_i^*)).$$

Hense φ is the isomorphism from $d-$ module X_m^* to $d-$ module X_m and bases $\{x_i\}$, $\{(-1)^{i+1}x_i\}$ are dual ones. $\square$

Definition 2.4. *Suppose $V = \{v_i\}$, $V^* = \{v_i^*\}$ are dual $D-$ modules in $k[X]$. The the polynomial*

$$\Delta(V, V^*) := \sum_i (v_i \cdot v_i^*).$$

is called the **Casimir element** *of a derivation D.*

By using theorem 2.1 we obtain following Casimir elements of degree 2 for the Weitzenbök derivation d:

$$\Delta(X_k, X_k^*) = \sum_{i=1}^{k}(-1)^{i+1}x_i \cdot x_{k-i+1}.$$

It is easy to show that $\Delta(X_k, X_k^*) \in k[X]^d$. In generally the following theorem holds:

Theorem 2.2. *Suppose U and U^* are two dual $D-$ modules in $k[X]$. Then $\Delta(U, U^*) \in k[X]^D$.*

Proof. Assume that the bases $\{u_i\}$, $\{u_i^*\}$ are dual ones and $D_U = \{\lambda_{ij}\}$, $i, j = 1 \ldots n$. Then

$$D(u_i) = \sum_{j=1}^{n} \lambda_{ij} u_j, D(u_i^*) = \sum_{j=1}^{n} (-\lambda_{ji} u_j).$$

Therefore

$$D(\Delta(U, U^*)) = D\Big(\sum_{i=1}^{n} u_i u_i^*\Big) = \sum_{i=1}^{n} (D(u_i) u_i^* + u_i D(u_i^*)) =$$

$$= \sum_{i=1}^{n} \Big(\sum_{j=1}^{n} (\lambda_{ij} u_j) u_i^* + u_i D(u_i^*)\Big) = \sum_{i=1}^{n} \Big(\sum_{j=1}^{n} \lambda_{ji} u_i u_j^* + u_i D(u_i^*)\Big) =$$

$$= \sum_{i=1}^{n} u_i \Big(\sum_{j=1}^{n} \lambda_{ji} u_j^* + D(u_i^*)\Big) = 0.$$

$\square$

For any Casimir element one can show that this element does not depend on the choice of dual bases; so that a Casimir element is well defined.

For linear derivation D of $k[X]$ a theorem inverse to theorem 2.2 is true. Let us assume that

$$D(x_i) = \sum_{j=0}^{n} \lambda_{ij} x_j.$$

Theorem 2.3. *Let z be homogeneous polynomial belonging to $k[X]^D$. Then z is a Casimir element.*

Proof. Let us remember that a derivation of the algebra $k[X]$ of a form $f_0 \partial_0 + f_1 \partial_1 + \cdots + f_n \partial_n$, $f_i \in k[X]$ is called a special derivation. It is well-known that the set of all special derivation $W_n := \mathrm{Der}(k[X])$ is a Lie algebra with respect to the commutator of a derivations. Taking into account $D = D(x_0)\partial_0 + D(x_1)\partial_1 + \cdots + D(x_n)\partial_n$ we get

$$[D, \partial_i] = -\sum_{i=0}^{n} \lambda_{ji} \partial_j.$$

It is clear that vector space X_n is $D-$module. Without loss of generality, we can assume that $\partial_n(z) \neq 0$.

Lema 2.1. *Let z be non-vanishing element from $k[X]^D$. Then the vector space $Z_D := \langle \partial_0(z), \cdots, \partial_n(z) \rangle$ is $D-$ module and $Z_D \cong X_n^*$.*

Proof. We need to verify duality of the bases elements now. In fact

$$D(\partial_i(z)) = [D, \partial_i](z) + \partial_i(D(z)) = [D, \partial_i](z) = -\sum_{i=0}^{n} \lambda_{ji} \partial_j(z).$$

$\square$

Now since the modules X_n and Z_D are dual we can write their Casimir element. By using Euler's theorem about homogeneous polynomials we get

$$\Delta(X_n, Z_D) = x_0 \partial_0(z) + x_1 \partial_1(z) + \cdots + x_n \partial_n(z) = \deg(z)\, z.$$

Thus $z = \dfrac{1}{\deg(z)} \Delta(X_n, Z_D)$, i.e., z is a Casimir element. $\square$

We end this section with the following

Conjecture. *For any derivation D the algebra $k[X]^D$ is generated by Casimir elements.*

3. Casimir elements of Weitzenбök derivation.

From theorem 2.3 it follow that to know the ring $k[X]^D$ we have to know a realisations of $D-$ modules in $k[X]$. Below we offer a way to construct these realisations for Weitzenбök derivation d.

Theorem 3.1. *Any $d-$ module $V = \langle v_0, v_1, \ldots v_n \rangle$ can be extended to $\mathfrak{sl}_2-$ module where $\mathfrak{sl}_2$ is simple three-dimensional Lie algebra over field k.*

Proof. Let us introduce on V two additional derivations $\hat{d}$ and e as follows:

$$\hat{d}(v_i) := (i+1)(n-i)v_{i+1},$$
$$e(v_i) := (n-2\,i)v_i.$$

By straightforward calculation for any i we get:

$$[d, \hat{d}](v_i) = e(v_i),$$
$$[d, e](v_i) = -2\,d(v_i),$$
$$[\hat{d}, e](v_i) = 2\,\hat{d}(v_i).$$

Those commutator relations coincide with the commutator relations of bases elements of the simple three-dimensional Lie algebra $\mathfrak{sl}_2$. Hence the vector space V_n together with operators $d, \hat{d}, e$ is $\mathfrak{sl}_2-$ module. $\qquad\square$

Definition 3.1. *For any polynomial $z \in k[X]$ a natural number s is called an **order** of the polynomial z if the number s is the smallest natural number such that*

$$\hat{d}^s(z) \neq 0, \hat{d}^{s+1}(z) = 0.$$

We denote an order of z by $\operatorname{ord}(z)$. For example $\operatorname{ord}(x_0) = n$. By using Leibniz's formula we get

$$\operatorname{ord}(a \cdot b) = \operatorname{ord}(a) + \operatorname{ord}(b) \text{ for all } a, b \in k[X].$$

For the derivation e every monomial $x_0^{\alpha_1} x_1^{\alpha_2} \cdots x_n^{\alpha_n}$ is an eigenvector with the eigenvalue $w(x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_n^{\alpha_n})$ where

$$w(x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_n^{\alpha_n}) = n\left(\sum_i \alpha_i\right) - 2\left(\alpha_1 + 2\,\alpha_2 + \cdots + n\,\alpha_n\right).$$

A homogeneous polynomial is called **isobaric** if all its monomials have equal eigenvalue.

Definition 3.2. *An eigenvalue of arbitrary monomial of a homogeneous isobaric polynomial z is called the **weight** of the polinomial z and denoted by $\omega(z)$.*

It is easy to see that $\omega(a \cdot b) = \omega(a) + \omega(b)$ for a homogeneous isobaric polynomials a, b.

Theorem 3.2. *For arbitrary homogeneous isobaric polynomial $z \in k[X]^d$ a vector space*

$$V_m(z):=\langle v_0, v_1, \ldots v_m \rangle, v_i = \frac{(w(z)-i)!}{i!\,(\omega(z))!}\,\hat{d}^i(z), v_0 := z, m = 0 \ldots s,$$

is $d-$ module, moreover $X_m \cong V_m(z)$. Here $\omega(z), s$ are weight and order of z.

Proof. Let us prove two intermediate relations :

(i) $e(\hat{d}^i(z)) = (\omega(z) - 2\,i)\,\hat{d}^i(z),$

(ii) $d(\hat{d}^i(z)) = i\,(\omega(z) - i + 1)\,\hat{d}^{i-1}(z).$

The relation (i) is valid for $i = 0$:

$$e(\hat{d}^0(z)) = e(z) = \omega(z)\, z.$$

If this relation holds for certain i, then

$$e(\hat{d}^{i+1}(z)) = [e, \hat{d}](\hat{d}^i(z)) + \hat{d}(e(\hat{d}^i(z))) = -2\,\hat{d}^{i+1}(z) + \hat{d}((\omega(z) - 2i)\hat{d}^{i+1}(z)) =$$
$$= (\omega(z) - 2(i+1))\,\hat{d}^{i+1}(z).$$

The relation (ii) is true for $i = 0$. In fact

$$d(\hat{d}^0(z)) = d(z) = 0$$

If this relation holds for certain i then by using (i) we get

$$d(\hat{d}^{i+1}(z)) = [d, \hat{d}](\hat{d}^i(z)) + \hat{d}(d(\hat{d}^i(z))) = e(\hat{d}^i(z)) + \hat{d}(i(\omega(z) - i + 1)\,\hat{d}^{i-1}(z)) =$$
$$= (\omega(z) - 2\,i)\hat{d}^i(z) + i(\omega(z) - i + 1)\hat{d}^i(z)) = (i+1)(\omega(z) - i)\,\hat{d}^i(z).$$

Hence the relations are valid for any i. Consider now a vector space

$$V_m(z) := \langle v_0, v_1, \cdots, v_m \rangle,$$

where $v_i = \alpha_i(z)\,\hat{d}^i(z)$ for some undefined $\alpha_i(z) \in k$. For the vector space $V_m(z)$ to be a $d-$ module it is enought to have $d(v_i) = v_{i-1}$ for all i. Since

$$d(v_i) = d(\alpha_i(z)\hat{d}^i(z)) = \alpha_i(z)\, i\,(\omega(z) - i)\,\hat{d}^{i-1}(z),$$

we get a following recurrence formula for $\alpha_i(z)$:

$$i\,(\omega(z) - i + 1)\,\alpha_i(z) = \alpha_{i-1}(z), \alpha_1(z) = 1,$$

by solving it we obtain

$$\alpha_i(z) = \frac{(\omega(z) - i)!}{i!\omega(z)!}.$$

$\square$

4. Maps τ_i.

For arbitrary homogeneous isobaric polynomial $z \in k[X]^d$ denote by $\tau_i(z)$ the Casimir element:

$$\tau_i(z) := \Delta(X_i, V_i^*(z)), 0 \le i \le \min(\mathrm{ord}(z), n), z \in k[X]^d.$$

Since every polynomial of $k[X]^d$ is a sum of a homogeneous isobaric polynomials, the map τ_i can be extended to whole algebra $k[X]^d$ by requiring $\tau_i(u+v) := \tau_i(v) + \tau_i(v)$. Besides we clearly have $\tau_i(\lambda z) = \lambda \tau_i(z), \lambda \in k$. Thus $\tau_i : k[X]^d \longrightarrow k[X]^d$ is now a linear map. Note that the maps τ_i is well defined only on elements z of a kernel such that $\mathrm{ord}(z) \le i$.

Theorem 4.1. *Any polynomial of $z \in k[X]^d$ has the form*

$$z = \frac{1}{\deg(z)}(\tau_n(c(0)) + \tau_{n-1}(c_1) + \ldots + \tau_0(c(n))),$$

where

$$c(0) = \partial_n(z),$$
$$c(i) = \partial_{n-i}(z) + \sum_{k=1}^{i} (-1)^{k+1} c_k(i-k),$$
$$c_k(i) := \frac{(w(c(i)) - k)!}{k!\, w(c(i))!}\, \hat{d}^k(c(i)), c_0(i) := c(i).$$

Proof. We may assume that z is a homogeneous isobaric polynomial. We first show that $d(c(i)) = 0$ for all i. Since $[\partial_n, d] = 0$ we have $d(c(0)) = d(\partial_n(z)) = \partial_n(d(z)) = 0$. Suppose by induction $d(c(i)) = 0$. We have to show that $d(c(i + 1)) = 0$. From theorem 3.2 it follows that $d(c_k(i)) = c_{k-1}(i)$. By definition we have

$$c(i + 1) = \partial_{n-(i+1)}(z) + \sum_{k=1}^{i+1}(-1)^{k+1}c_k(i + 1 - k) = \partial_{n-(i+1)}(z) + c_1(i) + \sum_{k=2}^{i+1}(-1)^{k+1}c_k(i + 1 - k).$$

Therefore

$$d(c(i + 1)) = d(\partial_{n-(i+1)}(z)) + d(\sum_{k=2}^{i+1}(-1)^{k+1}c_k(i + 1 - k)) + d(c_1(i)) =$$

$$= -\partial_{n-1}(z) + \sum_{k=2}^{i+1}(-1)^{k+1}c_{k-1}(i + 1 - k) + c(i) =$$

$$= -(\partial_{n-i}(z) + \sum_{k=1}^{i+1}(-1)^{k+1}c_k(i + 1 - k)) + c(i) = -c(i) + c(i) = 0.$$

Now since $\partial_n(z) = c(0)$ and $\mathrm{ord}(\partial_n(z)) = \mathrm{ord}(z) + n \geq n$ then the Casimir element $\tau_n(c(0))$ exists and

$$\tau_n(c(0)) = x_n c(0) - x_{n-1}\, c_1(0) + \ldots + (-1)^n x_0 c_n(0).$$

Furthermore, taking into account $\deg(z)z = x_n\partial_n(z) + x_{n-1}\partial_{n-1}(z) + \ldots + x_0\partial_0(z)$ we get :

$$\deg(z)z - \tau_n(c(0)) = x_{n-1}(\partial_n(z) + c_1(0)) + x_{n-2}(\partial_{n-2}(z) - c_2(0)) + \ldots + x_0(\partial_0 - (-1)^n c_n(0).$$

The coefficient of x_{n-1} is equal $c(1)$ hence

$$\tau_{n-1}(\partial_n(z) + c_1(0)) = x_{n-1}c_0(1) - x_{n-2}\, c_1(1) + \ldots + (-1)^n x_0 c_{n-1}(1).$$

Therefore

$$\deg(z)\, z - (\tau_n(c(0)) + \tau_{n-1}(c(1))) = x_{n-2}(\partial_{n-2}(z) - c_2(0) + c_1(1)) + \ldots +$$

$$+ x_i\, (\partial_i(z) - (-1)^i c_i(0) - (-1)^{i-1}c_{i-1}(1) + \ldots + x_0\, (\partial_0(z) - (-1)^n c_n(0) - (-1)^{n-1}c_{n-1}(1) =$$

$$= x_{n-2}(c(2)) + \ldots + x_0\, (\partial_0(z) - (-1)^n c_n(0) - (-1)^{n-1}c_{n-1}(1).$$

Finally, we obtain

$$\deg(z)\, z - (\tau_n(c(0)) + \tau_{n-1}(c(1)) + \ldots + \tau_1(c(n - 1)) =$$

$$= x_0\, (\partial_0(z) + c_1(n) - c_2(n - 1) + \ldots + (-1)^{n+1}c_n(0) = x_0\, c(n) = \tau_0(c(n)).$$

Hence

$$\deg(z)z = \tau_n(c(0)) + \tau_{n-1}(c(1)) + \ldots + \tau_1(c(n - 1)) + \tau_0(c(n)),$$

and we get

$$z = \frac{1}{\deg(z)}(\tau_n(c(0)) + \tau_{n-1}(c_1) + \ldots + \tau_0(c(n))).$$

$\square$

Theorem 4.2. *Let z be a homogeneous isobaric polynomial of $k[X]^d$. Then*
  (i) $\mathrm{ord}(\tau_i(z)) = n + \mathrm{ord}(z) - 2\, i,$
  (ii) $\omega(z) = \mathrm{ord}(z).$

Proof. (*i*) Let us denote by s the order of z. We first show that $\hat{d}^{n+s-2i}(\tau_i(z){\neq}0$ but $\hat{d}^{n+s-2i+1}(\tau_i(z){=}0$. Consider now two $\hat{d}-$modules:

$$\hat{X}_m := \langle x_n, \frac{x_{n-1}}{\gamma_{n-1}}, \frac{x_{n-2}}{\gamma_{n-1}\,\gamma_{n-2}}, \cdots \frac{x_m}{\gamma_{n-1}\,\gamma_{n-2}\cdots\gamma_{n-m}} \rangle,$$

$$\hat{V}_m(z) := \langle z_s, \frac{z_{s-1}}{\alpha_{s-1}(z)}, \frac{z_{s-2}}{\alpha_{s-1}(z)\,\alpha_{s-2}(z)} \cdots \frac{z_m}{\alpha_{s-1}(z)\,\alpha_{s-2}(z)\cdots\alpha_{s-m}(z)} \rangle,$$

where $\gamma_i{=}(i+1)\,(n-i)$, $z_i:=\alpha_i(z)\hat{d}^i(z)$, $z_0:=z$, $\alpha_i{=}\dfrac{(w(z)-i)!}{i!w(z)!}$, $m{=}0..\min(n,s)$ and $w(z)$ is a weight of the polynomial z. Define a linear multiplicative map $\psi : X_m \cdot V_m(z) \longrightarrow \hat{X}_m \cdot \hat{V}_m(z)$ by the rule

$$\psi(x_i) = \frac{x_{n-i}}{\gamma_{n-1}\,\gamma_{n-2}\cdots\gamma_{n-i}}, \psi(z_i) = \frac{z_{n-i}}{\alpha_{n-1}(z)\,\alpha_{n-2}(z)\ldots\alpha_{n-i}(z)}.$$

Since $\hat{x}_i = \gamma_i\,x_{i+1}$ and

$$\psi(d(x_i)) = \psi(x_{i-1}) = \frac{x_{n-(i-1)}}{\gamma_{n-1}\,\gamma_{n-2}\cdots\gamma_{n-(i-1)}},$$

$$\hat{d}(\psi(x_i)) = \hat{d}\left(\frac{x_{n-i}}{\gamma_{n-1}\,\gamma_{n-2}\cdots\gamma_{n-i}}\right) = \frac{x_{n-(i-1)}}{\gamma_{n-1}\,\gamma_{n-2}\cdots\gamma_{n-(i-1)}},$$

it follows that $\psi(d(x_i)){=}\hat{d}(\psi(x_i))$ thus the restriction of ψ to X_m is a isomorphism from $d-$module X_m to $\hat{d}-$ module $\hat{X}_m$. Similarly $V_m(z)$ and $\hat{V}_m(z)$ are isomorphic thus ψ is a isomorphism of $d-$ module $X_m \cdot V_m(z)$ to $\hat{d}-$ module $\hat{X}_m \cdot \hat{V}_m(z)$. From Theorem 3.2 it follows that $\hat{d}-$ modules $\hat{X}_m$ and $\hat{V}_m(z)$ are isomorphic. For arbitrary homogeneous isobaric polynomials $z \in k[X]^d$ denote by $\hat{\tau}_i(z)$ the corresponding Casimir element

$$\hat{\tau}_i(z) = \Delta(\hat{X}_i, \hat{V}_i(z)).$$

It is easy to check that $\hat{\tau}_i(z){=}\psi(\tau_i(z))$. Let us show that

$$\hat{d}^{n+s-2}(\tau_1(z)) = \frac{(n+s-2)!(n-1)!(s-1)!}{n\,s}\,\hat{\tau}_1(z).$$

Since

$$\gamma_0 \cdot \gamma_1 \cdots \gamma_{k-1} = (1 \cdot n)(2 \cdot (n-1))\ldots(k \cdot (n-(k-1))) = [k!]^2\binom{n}{k},$$

we have

$$\hat{d}^k(x_i) = \gamma_i\gamma_{i+1}\ldots\gamma_{i+k-1} \cdot x_{k+i} = \frac{\gamma_0\gamma_1\ldots\gamma_{i+k-1}}{\gamma_0\gamma_1\cdots\gamma_{i-1}}x_{k+i} = \frac{[(i+k)!]^2\binom{n}{k+i}}{[i!]^2\binom{n}{i}}x_{k+i}.$$

In particulary $\hat{d}^{n-1}(x_0){=}[(n-1)!]^2n \cdot x_{n-1}$, $\hat{d}^{n-1}(x_1){=}\dfrac{[(n)!]^2}{n} \cdot x_n$. Likewise,

$$\hat{d}^k(z_i) = \frac{[(i+k)!]^2\binom{s}{k+i}}{[i!]^2\binom{s}{i}}x_{z+i}.$$

Taking into account $\hat{d}^{n+1}(x_0){=}\hat{d}^{s+1}(z_0){=}0$ and $\tau_1(z) = x_0\,z_1 - x_1\,z_0$, we obtain

$$\hat{d}^{n+s-2}(x_0 \, z_1 - x_1 \, z_0) = \binom{n+s-2}{n} \hat{d}^n(x_0) \cdot \hat{d}^{s-2}(z_1) + \binom{n+s-2}{n-1} \hat{d}^{n-1}(x_0) \cdot \hat{d}^{s-1}(z_1)) -$$

$$- \binom{n+s-2}{n-1} \hat{d}^{n-1}(x_1) \cdot \hat{d}^{s-1}(z_0) - \binom{n+s-2}{n-2} \hat{d}^{n-2}(x_1) \cdot \hat{d}^s(z_0) =$$

$$= x_{n-1} z_s \left(\binom{n+s-2}{n-1} \cdot [(n-1)!]^2 n \frac{[(s-1)!]^2}{s} - \binom{n+2-2}{n-2} [(n-1)!]^2 [s!]^2 \right) +$$

$$+ x_n z_{s-1} \left(\binom{n+s-2}{n} [n!]^2 [(s-1)!]^2 - \binom{n+s-2}{n-1} \cdot [(s-1)!]^2 s \frac{[(n-1)!]^2}{n} \right) =$$

$$= (n+s-2)! \, s! \, (n-1)! \, x_{n-1} \, z_s - (n+s-2)! \, n! \, (s-1)! \, x_n z_{s-1} =$$

$$= (n+s-2)!(n-1)!(s-1)!(s \, x_{n-1} z_s - n \, x_n z_{s-1}) =$$

$$= \frac{(n+s-2)!(n-1)!(s-1)!}{n \, s} (\frac{x_{n-1}}{n} z_s - x_n \frac{z_{s-1}}{s}) = \frac{(n+s-2)!(n-1)!(s-1)!}{n \, s} \hat{\tau}_1(z).$$

In the general case, it can be shown (the proof is routine) that

$$\hat{d}^{n+s-2\,i}(\tau_i(z)) = \frac{(n+s-2\,i)!(n-i)!(s-i)!}{n(n-1)...(n-(i-1))s(s-1)...(s-(i-1))} \hat{\tau}_i(z) \neq 0,$$

whenever $\tau_i(z) \neq 0$, but $\hat{d}^{n+s-2i+1}(\tau_i(z)) = \hat{d}(\hat{d}^{n+s-2\,i}(\tau_i(z))) = 0$ since $\hat{\tau}_i(z)$ belongs to the kernel of the derivation $\hat{d}$, therefore the order of non-vanished polinomial $\tau_i(z)$ is equal $n+s-2\,i$.

(ii) It is a direct corollary of the relation $d(\hat{d}^i(z)) = i\,(\omega(z) - i + 1)\,\hat{d}^{i-1}(z)$.

$\square$

5. An algorithm of computing of the kernel $k[X]^d$

For any subalgebra $T \subseteq k[X]^d$ we write $\tau(T)$ for the subalgebra generated by the elements $\tau_i(z), z \in T, i \leq \mathrm{ord}(z)$. The map τ allow us to arrange the following iteration process for calculation of the algebra $k[X]^d$: for arbitrary subalgebra B of algebra $k[X]^d$ denote by $\overline{B}$ a subalgebra generated by elements $B \bigcup \tau(B)$. For every integer $m \geq 0$ define the following sequence B_m of subalgebras of $k[X]^d$

$$\begin{cases} B_0 = k[x_0] \\ B_m = \overline{B_{m-1}}. \end{cases}$$

We get a increasing chain of the subalgebras B_i

$$B_0 \subseteq B_1 \subseteq B_2 \ldots$$

Now it is possible to state the following proposition

Theorem 5.1. *There exists some k such that $B_k = B_{k+1} = k[X]^d$.*

This theorem is a direct corollary of finite generation of the Weitzenbök derivation and of the follow theorem :

Theorem 5.2. *If T is a subalgebra of $k[X]^d$ with the property that $x_0 \in T$ and $\tau(T) \subseteq T$, then $T = k[X]^d$.*

Proof. It is enought to show that $k[X]^d \subseteq T$. The proof it is by induction on the degree of a polynomial $z \in T$. Under the condition of the theorem we have $x_0 \in T$. Let us assume that the theorem is true for all polynomials of degree less or equal to s. Suppose $\deg(z) = s + 1$; then from theorem 4.1 it is follows that z can be expresed as a sum of polynomials $\tau_i(z_i')$ where z_i' are elements of the degree s. Then by the induction hypothesis, we have $z \in T$ therefore $k[X]^d \subseteq T$. □

For the realization of the algorithm it is necessary to be able to calculate algebra $\tau(B_i)$ knowing generating set for algebra B_i. Let us give a definition of a irreducible polynomials of algebra $k[X]^d$. We will say that x_0 is the only irreducible polynomial of depth unity, and that $x_0, K_1, \cdots, K_s$ form a complete set of irreducible polynomials of depth $< m$ if every polynomials of B_i, $i < m$ is a polynomial in $x_0, K_1, \ldots, K_s$, over field k. Polynomials which are not irreducible we will call as reducible. A several cases of reducebility considered in the following theorem.

Theorem 5.3. *Suppose that $u, v \in k[X]^d$ are irreducible polynomials of depth i; then*
   (i) *If $\operatorname{ord}(u) = 0$ then $\tau_k(u\,v)$ is reducible of the depth i;*
   (ii) *$\tau_i(u\,v)$ is reducible for all $i \leqslant \min(n, \operatorname{ord}(v))$;*
   (iii) *Polynomial $\tau_i(u_1\,u_2 \ldots u_{i+1})$, $u_i \in B_i$ is always reducible .*

Proof. (i) Let $i \leqslant \operatorname{ord}(v)$ and $V_i(v) = \langle v_0 v_1 \ldots, v_i \rangle$. If $\hat{d}(u) = 0$ then we have $V_i(u\,v) = \langle u\,v_0, u\,v_1 \ldots, u\,v_i \rangle$. Thus

$$\tau_i(u\,v) = \Delta(X_i, V_i^*(u\,v)) = \sum_{k=1}^{i}(-1)^{k+1} x_k\, u\, v_{i-k+1} = u\,\tau_i(v).$$

(ii) Let us show that for $i \leqslant \min(n, \operatorname{ord}(v))$ and for certain $c_k' \in k[X]^d$ the relation

$$\tau_i(u\,v) = u\,\tau_i(v) + \sum_{k=1}^{i-1}\tau_{i-k}(c_k'),$$

is true. In fact

$$\tau_i(u\,v) - u\,\tau_i(v) = \sum_{k=0}^{i}(-1)^{i-k}\alpha_{i-k}(u\,v)x_{i-k}\hat{d}^k(u\,v) - u\left(\sum_{k=0}^{i}(-1)^{i-k}\alpha_{i-k}(v)x_{i-k}\hat{d}^k(v)\right) =$$

$$= x_i\,uv + \sum_{k=1}^{i}(-1)^{i-k}\alpha_{i-k}(u\,v)x_{i-k}\hat{d}^k(u\,v) - u\left(x_i v + \sum_{k=1}^{i}(-1)^{i-k}\alpha_{i-k}(v)x_{i-k}\hat{d}^k(v)\right) =$$

$$= \sum_{k=1}^{i}(-1)^{i-k}x_{i-k}(\alpha_{i-k}(u\,v)\hat{d}^k(u\,v) - \alpha_{i-k}(v)u\,\hat{d}^k(v)$$

The polynomial of the right hand side belongs to the kernel as difference of two elements of the kernel. To conclude the proof, it remains to apply theorem 4.1

(iii) If among the polynomials $u_1, u_2, \ldots, u_{i+1}$ there is a polynomial of the order zero then $\tau_i(u_1\,u_2 \ldots u_{i+1})$ is reducible by (i). If all of them have non-vanishing orders then the order of $u_1\,u_2 \ldots u_i$ is $\geq i$ and $\tau_i(u_1\,u_2 \ldots u_{i+1})$ is reducible by (ii). □

Definition 5.1. *A polynomial $z \in \tau(B_m)$ is called* **acceptable** *for algebra B_m if z is ireducible and $z \notin B_m$.*

Let $B_m = k[f_1, \ldots, f_r, g_1, \ldots, g_s]$ where $\{g_i\}$ are acceptable for B_{m-1}. From theorem 4.3 it is follows

Theorem 5.4. *Polynomial* $\tau_i(f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_r^{\alpha_r} g_1^{\beta_1} \cdots g_s^{\beta_s})$ *can not be an acceptable polynomial for* B_m *if any of follow conditions holds :*

1. $\sum\limits_k \alpha_k + \sum\limits_k \beta_k > i.$
2. $\sum\limits_k \beta_k = 0.$
3. *Some of* f_i, g_k *have the order equal to zero but* $\alpha_i \neq 0, \beta_k \neq 0.$
4. $f_1^{\alpha_1} f_2^{\alpha_2} f_r^{\alpha_r} g_1^{\beta_1} g_s^{\beta_s}$ *can be expresed as product of two polynomials one of them has order greater than* $i.$

Definition 5.2. *A triple of integer numbers*

$$[\deg(z), \mathrm{ord}(z), \frac{n \deg(z) - \omega(z)}{2}]$$

is called the signature of a polynomial z *and denoted by* $[z]$.

From theorem 4.2 it is follow that $[u\,v] = [u] + [v]$
Now let us offer a verification algorith if a polinomial $z \in k[X]^d$ belongs to subalgebra $B = k[f_1, f_2, \ldots, f_m]$ of algebra $k[X]^d$.

(1) Set up the following system of equation

$$[z] = \alpha_1 [f_1] + \alpha_2 [f_2] + \cdots + \alpha_m [f_m]$$

(2) If this system has no an integer positive solutions then z doesnt belong to subalgebra B.
(3) Suppose $\{\alpha^{(i)} = (\alpha_1^{(i)}, \cdots, \alpha_m^{(i)}), i = 1..k\}$ is the set of all positive integer solutions of the system. Set up the new system of equations:

$$z = \beta_1 f^{\alpha^{(1)}} + \beta_2 f^{\alpha^{(2)}} + \cdots + \beta_k f^{\alpha^{(k)}},$$

where

$$f^{\alpha^{(i)}} = f_1^{\alpha_1^{(i)}} f_2^{\alpha_2^{(i)}} \cdots f_m^{\alpha_m^{(i)}}.$$

(4) If the system has non-vanishing solutions then $z \in B$ otherwise $z \notin B$

From above we obtain the following algorithm for computing of kernel $k[X]^d$. Let $\{B\}$ be a generating set of some algebra B.

(1) $\{B_1\} = \{x_0\}$.
(2) Suppose that algebra $\{B_i\} = \{f_1, \ldots, f_r\} \bigcup \{g_1, \ldots, g_s\}$ is already calculated where $\{g_1, \ldots, g_s\}$ are acceptable polynomials for B_i.
(3) Consider a finite sets of elements of $\tau(B_i)$ which could be an acceptable for B_i:

$$B_i^{(m)} := \{\tau_k(f_1^{\alpha_1} \cdots f_r^{\alpha_r} g_1^{\beta_1} \cdots g_s^{\beta_s}), \sum \alpha_q + \sum \beta_q = m, m \leq n, k \leq n\}$$

(4) By using previuos algoritm we compute a set H of acceptable polynomials of $B_i^{(m)}$, $m \leq n$.
(5) If $H = \emptyset$ then $k[X]^d = B_i$ else $B_{i+1} = \{f_1, \ldots, f_r\} \bigcup \{g_1, \ldots, g_s\} \bigcup H$.

6. A CALCULATION $k[X]^d$ FOR $n < 7$.

Denote by t the variable x_0.

Theorem 6.1. $B_1 = k[t, \tau_2(t), \tau_4(t), \cdots, \tau_{2[\frac{n}{2}]}(t)], n > 2$

Proof. It is clear that an aceptable polynomials for B_1 can only be any one of the following polynomials $\tau_i(t), i \leq n$. It is easy to check that for odd i we have $\tau_i(t) = 0$. To prove that the set of polynomials t, $\tau_2(t), \tau_4(t), \cdots, \tau_{2[\frac{n}{2}]}(t)$ is the minimal generating set for subalgebra B_1 it is enought to prove that there are not any linear relations for the polynomials t^2, $\tau_2(t), \tau_4(t), \cdots, \tau_{2[\frac{n}{2}]}(t)$. The proof follows obviously from the fact that no two polynomials have the same orders. $\qquad\square$

The following computations were all done with Maple.

5.1 $n = 1$.

Since $\tau_1(t) = 0$ we get $\tau(B_1) = 0$. Therefore, we have $B_2 = B_1$ thus $k[X]^d = k[t]$.

5.2 $n = 2$.

By using theorem 6.1 we have $B_1 = k[t, dv]$ where $dv := \tau_2(t) = t\,x_2 - 2x_1^2$. Since $\mathrm{ord}(dv) = 0$; then B_1 has no an acceptable elements. Therefore $B_2 = B_1$ and $k[X]^d = k[t, dv]$.

5.3 $n = 3$.

We have $B_1 = k[t, dv]$ where $dv = \tau_2(t)$. Since $\mathrm{ord}(dv) = 3 + 3 - 2 \cdot 2 = 2$, we obtain

$$B_1^{(1)} = \{\tau_1(dv), \tau_2(dv)\},$$
$$B_1^{(2)} = \{\tau_3(dv^2)\}.$$

However by a straightforward calculation we obtain $\tau_3(dv^2) = 0$ and $\tau_2(dv) = 0$. Denote the remaining element by $tr := \tau_1(dv)$. By using an algorithm of the section 5 we get $tr \notin B_1$ and thus $B_2 = k[t, dv, tr]$. Since $\mathrm{ord}(tr) = 3$ we see that an acceptable elements for B_2 can only be the polynomial $c = \tau_3(tr)$. Note $\mathrm{ord}(c) = 0$ and B_2 has no an elements of order zero. Hence $c \notin B_2$ and $B_3 = k[t, dv, tr, c]$. Since $\mathrm{ord}(c) = 0$, then B_3 has no any acceptable elements. Hense $B_4 = B_3$ thus $k[X]^d = k[t, dv, tr, c]$ where

$$dv = -2\,t\,x_2 + x_1{}^2,$$
$$tr = -3\,t\,x_1\,x_2 + 3\,t^2\,x_3 + x_1{}^3,$$
$$c = -18\,t\,x_1\,x_2\,x_3 + 8\,t\,x_2{}^3 + 9\,x_3{}^2\,t^2 + 6\,x_1{}^3\,x_3 - 3\,x_1{}^2\,x_2{}^2.$$

5.4 $n = 4$.

$B_1 = k[t, d_1, d_2]$, where

$$d_1 = \tau_2(t) \quad [d_1] = [2, 4, 4],$$
$$d_2 = \tau_4(t) \quad [d_2] = [2, 0, 12].$$

It is easy to see that for subalgebra B_1 only polynomial $\tau_i(d_1)$, $i = 1..4$ can be an acceptable polynomial. By direct calculation we obtain $\tau_2(d_1) = t\,d_2 \quad \tau_3(d_1) = 0$. Put

$$tr_1 = \tau_1(d_1) \quad [tr_1] = [3, 6, 8],$$
$$tr_2 = \tau_4(d_1) \quad [tr_2] = [3, 0, 18].$$

The signatures of t^3, $t\,d_1$, $t\,d_2$ are equal $[3, 8, 0]$, $[3, 8, 4]$, $[3, 4, 12]$ therefore tr_1, tr_2 are not in B_1, hence $B_2 = k[t, d_1, d_2, tr_1, tr_2]$. Since $\mathrm{ord}(d_2) = \mathrm{ord}(tr_2) = 0$ we see that an acceptable elements can only be the follow elements $\tau_i(tr_1)$, $i = 1..4$. Take into account $\tau_2(tr_1) = 0$, $\tau_2(tr_4) = 0$ and

$$\tau_1(tr_1) = d_2\,t^2 + d_1^2,$$
$$\tau_3(tr_2) = d_1\,d_2 - t \cdot tr_2,$$

we have $\tau(B_2) \subseteq B_2$, $B_3 = B_2$ and $k[X]^d = k[t, d_1, d_2, tr_1, tr_2]$ where

$$d_1 = -2\,t\,x_2 + x_1{}^2,$$
$$d_2 = -2\,t\,x_4 + 2\,x_1\,x_3 - x_2{}^2,$$
$$tr_1 = -3\,t\,x_1\,x_2 + 3\,x_3\,t^2 + x_1{}^3,$$
$$tr_2 = 12\,x_2\,t\,x_4 - 9\,x_3{}^2\,t - 6\,x_1{}^2\,x_4 + 6\,x_1\,x_2\,x_3 - 2\,x_2{}^3.$$

5.5 n = 6.

We have $B_2 = k[t, d_1, d_2]$ where

$$d_1 := \tau_2(t), [d_1] = [2, 6, 2],$$
$$d_2 := \tau_4(t), [d_2] = [2, 2, 4].$$

The following 10 polynomials can be acceptable polynomials for B_2 :

$$B_2^{(1)} = \{\tau_i(d_1), i = 1..5; \tau_i(d_2), i = 1, 2\},$$
$$B_2^{(2)} = \{\tau_i(d_2^2), i = 3, 4\},$$
$$B_2^{(3)} = \{\tau_5(d_2^3)\}.$$

By direct calculation we obtain

$$\tau_2(d_1) = -\tfrac{6}{5}\,t\,d_2, \quad \tau_5(d_1) = 0$$
$$\tau(d_2) = 5\,\tau_3(d_1) \quad \tau_2(d_2) = -\tfrac{5}{4}\tau_4(d_1).$$

Make the denotations,

$$tr_1 := \tau_4(d_1), \quad [tr_1] = [3, 3, 6],$$
$$tr_2 := \tau_3(d_1), \quad [tr_2] = [3, 5, 5],$$
$$tr_3 := \tau_1(d_1), \quad [tr_3] = [3, 9, 3],$$
$$p_1 := \tau_4(d_2^2), \quad [p_1] = [5, 1, 12],$$
$$p_2 := \tau_3(d_2^2), \quad [p_2] = [5, 3, 11],$$
$$si_1 := \tau_5(d_2^3), \quad [si_1] = [7, 1, 17].$$

Thus

$$B_3 = k[t, d_1, d_2, tr_{1-3}, p_{1-2}, si_1].$$

By using an algorithm of section 5 one can show that this polynomial system is the generating set for B_3 Now write down polynomials which can be acceptable for B_2:

$$B_2^{(1)} = \{\tau_3(tr_1), \tau_5(tr_2), \tau_4(tr_3), \tau_5(tr_3)\}$$
$$B_2^{(2)} = \{\tau_{4-5}(d_2\,tr_1), \tau_5(d_2\,p_2), \tau_5(tr_1^2)\}$$
$$B_2^{(3)} = \{\tau_5(d_2^2\,p_1), \tau_5(d_2^2\,tr_1), \tau_5(d_2^2\,si_1), \tau_5(tr_1\,p_1\,si_1), \tau_5(p_1\,p_2\,si_1)\}$$
$$B_2^{(4)} = \{\emptyset\}$$

By direct calculation we obtain that the following polynomials are equal to zero : $\tau_2(tr_1)$, $\tau_5(d_2\,tr_1), \tau_5(tr_1^2), \tau_5(d_2^2\,p_1), \tau_5(tr_1\,p_1\,si_1)$. Put

$$
\begin{aligned}
c_1 &:= \tau_5(tr_2) & [c_1] &= [4, 0, 10], \\
c_2 &:= \tau_4(tr_3) & [c_2] &= [4, 4, 8], \\
c_3 &:= \tau_5(tr_3) & [c_3] &= [4, 6, 7], \\
s_1 &:= \tau_4(d_2\,tr_1) & [s_1] &= [6, 2, 14], \\
v_1 &:= \tau_5(d_2\,p_2) & [v_1] &= [8, 0, 20], \\
v_2 &:= \tau_5(d_2^2\,tr_1) & [si_1] &= [8, 2, 19] \\
dv &:= \tau_5(dv_2^2\,si_1) & [dv] &= [12, 0, 30] \\
vis &:= \tau_5(p_1\,p_2\,si_1) & [vis] &= [18, 0, 45].
\end{aligned}
$$

Thus

$$
B_3 = k[t, d_1, d_2, tr_1, tr_2, tr_3, c_1, c_2, c_3, p_1, p_2, s_1, si_1, v_1, v_2, dv, vis]
$$

The following polynomials are acceptable for B_3:

$$
\begin{aligned}
p_3 &:= \tau_2(c_3) & [p_3] &= [5, 7, 9], \\
s_2 &:= \tau_1(p_1) & [s_2] &= [6, 4, 13], \\
si_2 &:= \tau_1(s_1) & [si_2] &= [7, 5, 15], \\
dev &:= \tau_2(v_2) & [dev] &= [9, 3, 21], \\
od &:= \tau_5(s_1\,d_2^2) & [od] &= [11, 1, 27], \\
trn &:= \tau_5(v_2\,d_2^2) & [trn] &= [13, 1, 32].
\end{aligned}
$$

As above we can show

$$
B_4 = k[t, d_{1-2}, tr_{1-3}, c_{1-3}, p_{1-3}, s_{1-2}, si_{1-2}, v_{1-2}, dev, od, dv, trn, vis],
$$

Similarly we obtain $\tau(B_4) \subset B_4$ hence the indicated polynomial set of 23 polynomials is a minimal generating set for $k[X]^d$.

5.6 n = 6. We have $B_2 = k[t, d_1, d_2, d_3]$ where

$$
\begin{aligned}
d_1 &:= \tau_6(t) & [p_3] &= [2, 0, 6], \\
d_2 &:= \tau_4(t) & [s_2] &= [2, 4, 4], \\
d_3 &:= \tau_2(t) & [si_2] &= [2, 8, 2],
\end{aligned}
$$

Further $B_3 = k[t, d_{1-3}, tr_{1-4}, p_{1-2}]$ where

$$
\begin{aligned}
tr_1 &:= \tau_6(d_3) & [p_3] &= [3, 2, 8], \\
tr_2 &:= \tau_4(d_3) & [s_2] &= [3, 6, 6], \\
tr_3 &:= \tau_3(d_3) & [si_2] &= [3, 8, 6], \\
tr_4 &:= \tau_1(d_3) & [si_2] &= [3, 12, 3], \\
p_1 &:= \tau_6(d_2^2) & [p_1] &= [5, 2, 14], \\
p_2 &:= \tau_5(d_2^2) & [p_1] &= [5, 4, 13].
\end{aligned}
$$

In the same way we obtain $B_4 = k[t, d_{1-3}, tr_{1-4}, c_{1-4}, p_{1-2}, s_{1-3}, si_{1-2}, vi, dev, de_{1-2}, dvan]$ where

$$
\begin{aligned}
c_1 &:= \tau_6(tr_2) & [c_1] &= [4, 0, 12], \\
c_2 &:= \tau_5(tr_3) & [c_2] &= [4, 4, 10], \\
c_3 &:= \tau_6(tr_4) & [c_3] &= [4, 6, 9], \\
c_4 &:= \tau_4(tr_4) & [c_4] &= [4, 10, 7], \\
s_1 &:= \tau_6(d_2\, tr_1) & [p_1] &= [6, 0, 18], \\
s_2 &:= \tau_3(d_2\, tr_1) & [p_1] &= [6, 6, 15], \\
s_3 &:= \tau_1(p_1) & [p_1] &= [6, 6, 15], \\
si_1 &:= \tau_4(tr_1^2) & [si_1] &= [7, 2, 20], \\
si_2 &:= \tau_3(tr_1^2) & [si_2] &= [7, 4, 19], \\
vi &:= \tau_6(d_2\, p_2) & [vi] &= [8, 2, 23], \\
dev &:= \tau_4(tr_1 p_2) & [dev] &= [9, 4, 25], \\
de_1 &:= \tau_6(tr_1^3) & [de_1] &= [10, 0, 30], \\
de_2 &:= \tau_5(tr_1^3) & [de_2] &= [10, 2, 29], \\
dvan &:= \tau_5(tr_1^2 p_1) & [dvan] &= [12, 2, 35].
\end{aligned}
$$

Acceptable polynomials for B_3 are only these two polynomials :

$$
\begin{aligned}
p_3 &:= \tau_4(c_4) & [p_3] &= [5, 8, 11], \\
pt &:= \tau_6(si_1\, si_2) & [pt] &= [15, 0, 45].
\end{aligned}
$$

The polynomial pt has degree 15 order 0 and consist of 1370 terms. A weight of pt is an odd number whereas the weights of all other generating polynomials of weight zero are even numbers. Therefore pt is irreducible. We can show that

$$B_4 = k[t, d_{1-3}, tr_{1-4}, c_{1-4}, p_{1-3}, s_{1-3}, si_{1-2}, vi, dev, de_{1-2}, dvan, pt].$$

As above we may obtain $\tau(B_4) \subset B_4$ hence the indicated polynomial set of 26 polynomials is a minimal generating set for $k[X]^d$.

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